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STABILITY OF CONVECTIVE FLOWS IN A ROTATING LIQUID LAYER UNDER VARIOUS HEATING CONDITIONS

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For a rotating liquid layer with boundaries of low thermal conductivity, an amplitude equation is obtained that describes the evolution of secondary convective flows in uniform heating and above a hot spot. The dependence of the coefficients of the amplitude equation on the rotation parameter, Prandtl number, and heat-flux nonuniformity is obtained. The influence of rotation on the stability of nonlinear regimes is analyzed for uniform heating. The boundaries of flow stability are investigated for variously shaped hot spots.

In a horizontal liquid layer with fixed boundaries of low thermal conductivity, instability of the equilibrium of the liquid under uniform heating is associated with longwave perturbations [1]. Nonlinear, steady, spatially periodic, two-dimensional convection regimes in such a layer are considered in [2] for small values of supercritical heating. Investigation of flows that arise is based on expansion in the small parameter ε, which is the ratio of the layer thickness h to the characteristic horizontal size of convective structures L_* ($\varepsilon = h/L_*$). In the case of nonuniform heating, equilibrium of the liquid is impossible, and flow arises. If the horizontal scale of heating nonuniformity (a hot spot) is large, the scale of the induced flows is also large. Hence, it is possible to conduct investigations in the longwave limit using expansion in the small parameter ε. The stability of convective flows induced by nonuniform heating is investigated by Lyubimov and Cherepanov [3].

Rotation of a liquid layer gives rise to convective instability of equilibrium of the liquid by a shortwave mechanism [4]. Because of competition of the two mechanisms, in the case of fast rotation of a liquid layer with heat-insulated boundaries, cellular perturbations become dangerous, and at rather low rotation velocities, longwave instability is realized [5].

In the present paper, we study the stability of longwave convective flows in a rotating horizontal liquid layer with boundaries of low thermal conductivity under uniform and nonuniform heating.

1. Formulation of the Problem. Case of Uniform Heating. Let a horizontal liquid layer of density ρ_0 rotate at constant angular velocity Ω about a vertical axis. On the boundaries of the layer, we specify a stationary uniform heat flux $Q = æ\,\partial T/\partial z$, where æ is the thermal conductivity of the liquid. We study the occurrence of convection in a coordinate system attached to the layer. The z axis of the Cartesian system is directed vertically upward, the coordinates of the boundaries are $z = \pm h/2$, and the x and y axes are located in the plane of the layer.

Centrifugal convective forces are ignored in comparison with gravitational forces. This is justified in the case where the horizontal scale of convective structures satisfies the condition

$$L \ll L_0 = g/\Omega^2, \qquad (1.1)$$

where g is the free-fall acceleration.

For the atmosphere, $\Omega \sim 7 \cdot 10^{-5}$ sec^{-1} and $L_0 \sim 2 \cdot 10^6$ km, while the size of a tropical cyclone is $L \sim 1500$ km [6]. In experimental modeling of large-scale eddies [7], $L_0 \sim 80$ m corresponds to the highest rotation frequency $\Omega \sim 0.4$ sec^{-1}, and the size of the model and the observed structure is $L \sim 0.3$ m. Thus, in

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the atmosphere and in experiments modeling atmospheric phenomena, large-scale structures satisfy relation (1.1).

Let us consider weakly nonlinear convection regimes that result from the evolution of convective waves [the flow characteristics do not depend on y ($\partial/\partial y = 0$)]. In the case of fixed heat-conducting boundaries, supercritical convective regimes in the form of billows [8] can occur far from the lateral walls of a laboratory model. On the other hand, in a layer with a free upper boundary and a fixed lower boundary of low thermal conductivity, the linear theory of convective stability for billows [9] and the experiment of [10] are in good agreement.

We use h, h^2/χ, χ/h, χ, and $Qh/æ$ as units of length, time, velocity, stream function, and temperature, respectively, and write the convection equation in a rotating coordinate system in dimensionless form:

$$\frac{1}{\mathrm{Pr}}\frac{\partial\Delta\Psi}{\partial t} + \frac{1}{\mathrm{Pr}}\Big[\frac{\partial\Psi}{\partial z}\frac{\partial\Delta\Psi}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial\Delta\Psi}{\partial z}\Big] = \Delta^2\Psi - \mathrm{Ra}\frac{\partial T}{\partial x} + D\frac{\partial v}{\partial z},$$

$$\frac{1}{\mathrm{Pr}}\frac{\partial v}{\partial t} + \frac{1}{\mathrm{Pr}}\Big[\frac{\partial\Psi}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial v}{\partial z}\Big] = \Delta v - D\frac{\partial\Psi}{\partial z}, \quad \frac{\partial T}{\partial t} + \frac{\partial\Psi}{\partial x} + \Big[\frac{\partial\Psi}{\partial z}\frac{\partial T}{\partial x} - \frac{\partial\Psi}{\partial x}\frac{\partial T}{\partial z}\Big] = \Delta T, \tag{1.2}$$

$$\mathrm{Ra} = g\beta Qh^4/\nu\chi æ, \quad D = 2\Omega h^2/\nu, \quad \mathrm{Pr} = \nu/\chi.$$

Here Ψ is the stream function, v is the liquid velocity along the y axis, T is the deviation of the temperature from the equilibrium value, Ra is the Rayleigh number, Pr is the Prandtl number, D is a parameter that describes the rotation velocity of the liquid, β, χ, and ν are the thermal-expansion coefficient, thermal diffusivity, and kinematic viscosity of the liquid.

The boundary conditions are of the form

$$z = \pm\frac{1}{2}: \qquad \Psi = 0, \quad \frac{\partial\Psi}{\partial z} = 0, \quad v = 0, \quad \frac{\partial T}{\partial z} = 0. \tag{1.3}$$

According to the linear theory of the stability of the equilibrium of a rotating liquid layer with heat-insulated boundaries [5], the rotation velocity of the layer determines the type of critical perturbation and the convection threshold $\mathrm{Ra}_0 = \mathrm{Ra}_0(D)$. In the case of rather slow rotation ($D < D_* \cong 43$), longwave perturbations increase. In fast rotation, instability is related to cellular perturbations. In the longwave case at $\mathrm{Ra} > \mathrm{Ra}_0$, perturbations with wave numbers in the interval $[0; k_*]$ increase. Here $k_* = 2\pi/L_*$ is the wave number that corresponds to the critical value of Ra. For small values of $\mathrm{Ra} - \mathrm{Ra}_0$, the value of $k_* \sim (\mathrm{Ra} - \mathrm{Ra}_0)^{1/2}$ is small, and longwave asymptotic relations can be used. In this case, if the scale $L_* \ll L_0$ of (1.1), the centrifugal force in the flow equation can be ignored.

We change the horizontal scale using the small parameter $\varepsilon = 1/L_*$:

$$x_n = \varepsilon x, \qquad \frac{\partial}{\partial x} = \varepsilon\frac{\partial}{\partial x_n}. \tag{1.4}$$

Below, the subscript n of the horizontal coordinate is dropped.

We study processes of various temporal scales using the method of multiple scales [11]. The functions Ψ, v, and T are assumed to depend on the set of variables $t_n = \varepsilon^n t$ ($n = 0, 1, 2, \ldots$). In this case, the derivative with respect to time is

$$\frac{\partial}{\partial t} = \sum_{n=0}^{\infty}\varepsilon^n\frac{\partial}{\partial t_n}. \tag{1.5}$$

It can be shown that the differentiation operator, as in [2, 3], begins with the fourth order of smallness. We expand Ψ, v, T, and the parameter Ra in series in ε:

$$\mathrm{Ra} = \sum_{n=0}^{\infty}\varepsilon^n\mathrm{Ra}_{2n}, \quad \Psi = \sum_{n=0}^{\infty}\varepsilon^n\Psi_n, \quad v = \sum_{n=0}^{\infty}\varepsilon^n v_n, \quad T = \sum_{n=0}^{\infty}\varepsilon^n T_n. \tag{1.6}$$

Substituting series (1.4)–(1.6) into (1.2) and (1.3), we obtain systems of differential equations in various orders of expansion in ε. We require boundedness of solutions as $t \to \infty$ and $x \to \pm\infty$.

In zero order, we have $\Psi_0 = 0$ and $v_0 = 0$, and the temperature $T_0(x)$ does not depend on the transverse coordinate z. In the first order of perturbation, the temperatures are also independent of z: $T_1 = T_1(x)$, and for Ψ_1 and v_1 we have

$$\Psi_1 = -\mathrm{Ra}_0 T_{0x} f_1(z), \qquad v_1 = -\mathrm{Ra}_0 T_{0x} f_2(z), \tag{1.7}$$

where

$$f_1(z) = a(\cos\lambda_1 z - \cos\lambda_1/2) + b(\cos\lambda_2 z - \cos\lambda_2/2);$$

$$f_2 = a(D/\lambda_1)\sin\lambda_1 z + b(D/\lambda_2)\sin\lambda_2 z - z/D; \quad a = i\lambda_2/[4D^2\sin(\lambda_1/2)]; \quad b = a^*;$$

$\lambda_{1,2} = \sqrt{\pm iD}$ are roots of the equation $\lambda^4 + D^2 = 0$; the asterisk and the subscript x denote complex conjugation and a derivative with respect to the horizontal.

In second order, we have

$$\Psi_2 = -\mathrm{Ra}_0 T_{1x} f_1(z), \qquad v_2 = -\mathrm{Ra}_0 T_{1x} f_2(z),$$

$$T_2 = -T_{0xx}\Big[\frac{z^2}{2}(1+\mathrm{Ra}_0 e) - \mathrm{Ra}_0\Big(\frac{a\cos\lambda_1 z}{\lambda_1^2} + \frac{b\cos\lambda_2 z}{\lambda_2^2}\Big)\Big] - \mathrm{Ra}_0(T_{0x})^2\Big[\frac{a\sin\lambda_1 z}{\lambda_1} + \frac{b\sin\lambda_2 z}{\lambda_2} + ez\Big] + \theta_2(x). \tag{1.8}$$

Here $e = -\mathrm{Im}(\lambda_1\cot(\lambda_2/2))/2D^2$ (Im is the imaginary part). Satisfying the boundary conditions for the temperature, we write

$$\mathrm{Ra}_0 = D^2 R_d, \quad R_d = 2/[\mathrm{Im}(\lambda_1\cot(\lambda_2/2)) - 2]. \tag{1.9}$$

The equations for Ψ_3, v_3, T_3 are cumbersome and are not given here. In fourth order we obtain the following differential equation for the temperature:

$$T_4'' = -T_{2xx} + \Psi_{3x} + \Psi_2' T_{1x} + \Psi_1' T_{2x} - \Psi_{1x} T_2' + \frac{\partial T_0}{\partial t_4}.$$

Integrating this equation across the layer yields the following equation for the evolution of perturbations for T_0:

$$\frac{\partial T_0}{\partial t} + A\frac{\partial^4 T_0}{\partial x^4} + \frac{\mathrm{Ra}_2}{\mathrm{Ra}_0}\frac{\partial^2 T_0}{\partial x^2} - B\frac{\partial}{\partial x}\Big(\Big(\frac{\partial T_0}{\partial x}\Big)^3\Big) = 0. \tag{1.10}$$

Linearizing Eq. (1.10), for neutral perturbations we obtain $\mathrm{Ra}_2 = A\mathrm{Ra}_0$.

The evenness of the eigenfunctions of the problem in various orders of expansion leads to the fact that the coefficients A and B depend only on the rotation parameter D:

$$A = -\frac{(7+5R_d)(\sin d - \sinh d)}{64\, d^5(\cosh d - \cos d)} - \frac{(3+R_d)(1-\cos d\cosh d)}{32\, d^4(1 - 2\cos d\cosh d + 0.5(\cos 2d + \cosh 2d))} - \frac{R_d + 1}{48\, d^4},$$

$$B = \frac{d^2}{4}R_d^2\Big(\frac{3\sin d\sinh d}{(\cosh d - \cos d)^2} - \frac{5(\sin d + \sinh d)}{2d(\cosh d - \cos d)} + \frac{\cosh d + \cos d}{\cosh d - \cos d}\Big).$$

Here $d = (D/2)^{1/2}$.

For $D = 0$, the values of the coefficients A and B are determined in [2]: $A = A_0 = 17/462$, $B = 10/7$, and $\mathrm{Ra}_2 = 2040/77$. With increase in the rotation velocity D, the coefficient A and, hence, Ra_2 decrease monotonically, reaching zero for $D = D_* = 43.5$, and the coefficient of the nonlinear term B decreases slowly, changing in the interval of interest ($D < D_*$) by 4%. The value of D_* obtained by numerical analysis of the stability of the equilibrium is somewhat lower: $D_* = 42$ [5]. To construct the amplitude equation that describes secondary convective flows for $D = D_*$, it is necessary to consider higher-order expansions in ε. We shall study only the case $D < D_*$ and use Eq. (1.10), which, in the new scales $t = \tau/A$ and $T_0 = \vartheta(A/B)^{1/2}$

is written as

$$\frac{\partial\vartheta}{\partial t}+\frac{\partial^4\vartheta}{\partial x^4}+\frac{\partial^2\vartheta}{\partial x^2}-\frac{\partial}{\partial x}\left(\left(\frac{\partial\vartheta}{\partial x}\right)^3\right)=0. \tag{1.11}$$

According to (1.7), the derivative $\partial\vartheta/\partial x \equiv N$ defines the convective flow intensity. In the steady case, $\partial/\partial\tau = 0$, and instead of (1.11) we obtain

$$\frac{\partial^3 N}{\partial x^3}+\frac{\partial N}{\partial x}-\frac{\partial N^3}{\partial x}=0. \tag{1.12}$$

Using N_x as an integrating factor, we write solution (1.12) in the form of a Jacobi elliptic function:

$$N=\left(\frac{2s^2}{1+s^2}\right)^{1/2}\sinh[(1+s^2)^{-1/2}x]. \tag{1.13}$$

The modulus of the elliptic function s is found using the periodicity condition in the absence of a mean horizontal heat flux:

$$N(x+L)=N(x), \qquad \int\limits_0^L N\,dx=0.$$

It is related to the size of convective structures L by the relation $L=4K(s)\sqrt{1+s^2}$, where $K(s)$ is a complete elliptic integral of the first kind. An increase in the convective-cell size L is accompanied by an increase in the amplitude of the convective flow N: $N\to 1$ as $L\to\infty$.

An analysis of the stability of two-dimensional, spatially periodic, secondary flows in a motionless layer with heat-insulated boundaries against normal perturbations of the form $\tilde{N}(x)\exp(-\alpha_0\tau)$ [2] has shown that all such flows are unstable: $\alpha_0<0$. Here α_0 is the increment of perturbations in a liquid at rest. The equation for the evolution of perturbations in a rotating layer can be reduced to the case of rest by transformation of scales. Taking into account that $t=\tau/A_0$ for $D=0$ and $t=\tau/A$ for $D\neq 0$, we obtain the following relation between the perturbation increments for rotating and motionless liquid layers: $\alpha=\alpha_0 A_0/A$. Since the coefficient A in the longwave region is positive, all two-dimensional spatially periodic flows of type (1.13) in a rotating layer are unstable ($\alpha<0$).

2. Case of Nonuniform Heating. We examine the case of weakly nonuniform heating. The nonuniformity of heating is considered a second-order infinitesimal. This yields closed equations for the evolution of temperature perturbations ϑ. The boundary conditions for the temperature can be rewritten as $z=\pm 1/2$: $\partial T/\partial z=\varepsilon^2 q(x)$, where $q(x)$ is the deviation of the heat flux from the average value measured in units of Q. A value $q<0$ corresponds to a heat flux above the critical value in the case of uniform heating. The change in the boundary conditions leads to the appearance of the additional term (T_2+qz) for second-order temperature perturbations in (1.8). As a result, the type of equation for ϑ changes:

$$\frac{\partial\vartheta}{\partial t}+\frac{\partial^4\vartheta}{\partial x^4}-\frac{\partial}{\partial x}\left(\left(\frac{\partial\vartheta}{\partial x}\right)^3+q(x)\frac{\partial\vartheta}{\partial x}\right)=0. \tag{2.1}$$

We examine the stability boundary of perturbations ($\partial/\partial t = 0$) for various types of heating nonuniformity. Integrating the linearized equation (2.1) and taking into account the attenuation of N at infinity, we obtain

$$\frac{\partial^2 N}{\partial x^2}-q(x)N=0. \tag{2.2}$$

Let the function $q(x)$ have the form of a step and the heat flux exceed the critical value in a bounded region:

$$q(x)=\left[\begin{array}{ll}-\gamma^2, & |x|<l,\\ \gamma^2, & |x|>l.\end{array}\right.$$

The solution of Eq. (2.2) for the given hot spot is of the form

$$N = c_1 \cos \gamma x (x < l), \quad N = c_2 \exp(-\gamma x)(x > l).$$

Requiring continuity of N and $\partial N/\partial x$ on the boundary of the hot spot ($x = l$), we determine the stability boundaries in the plane of the parameters l and γ: $\gamma l = \pi/4 + \pi n$ ($n = 0, 1, 2\ldots$). The stability region lies between the first hyperbola ($n = 0$) and the axis $\gamma = 0$. The lower the degree of superheating γ, the larger the size of the hot spot l that causes instability.

The equation for the horizontal heat flux N (2.2) with a square hot spot $q(x) = q_0(x^2 - 1)$ ($q_0 > 0$) reduces to the problem of a quantum-mechanical oscillator. The eigenvalue of the parameter q_0 that corresponds to the first level of instability is equal to unity. For $q_0 < 1$, the solution related to heat-flux nonuniformity is steady and $\vartheta \to 0$. The solution that describes the horizontal heat flux N on the stability boundary ($q_0 = 1$) is of the form $N = \exp(-x^2/2)$.

For the hot spot $q(x) = \gamma^2(\sinh^2\gamma x - 1)/(\cosh^2\gamma x)$, which has a minimum at $x = 0$ and is damped as γ^2 at infinity, the solution can be found using an analogy with the Schrödinger equation with a modified Peshl–Teller potential [12]. For the quantity N, we have solutions of the soliton type:

$$N = C/(\cosh\gamma x). \tag{2.3}$$

In this case, $\vartheta(x) = (C/\gamma)\arctan\sinh\gamma x$. The amplitude of the soliton C is an arbitrary constant.

The nonlinear stationary equation for the horizontal heat flux $\partial^2 N/\partial x^2 - q(x)N - N^3 = 0$ also has the soliton solution (2.3), but the form of the hot spot must be different: $q(x) = (\gamma^2\sinh^2\gamma x - h^2)/(\cosh^2\gamma x)$. In this case, the amplitude of the soliton (2.3) is completely determined by $C = \sqrt{h^2 - \gamma^2}$.

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